INVARIANT CROSS-SECTIONS AND INVARIANT LINEAR SUBSPACES*

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ABSTRACT

Existence theorems for lineaı subspaces invariant under a continuous mapping and contained in a given set are obtained from a gcneral theorem on existence ot invariant crosss-sections.

1. **Introduction.** For a continuous mapping π from a topological space X onto a topological space Y, a *cross-section* or a *π-cross-section* is, as usual, a continuous mapping ξ from Y into X such that $\pi \circ \xi$ is the identity mapping on Y. Together with π, let a set Γ of π-cross-sections and a continuous mapping ϕ from X into itself be given. We are interested in conditions which will ensure the existence of a $\xi \in \Gamma$ invariant under ϕ, i.e., $(\phi \circ \xi)(Y) \subset \xi(Y)$. In §2 we give such an existence theorem. In the extreme case when Y consists of a single point, a π-cross-section invariant under ϕ is of course a fixed point of ϕ. Thus Theorem 1 may be regarded as a new generalization of Tychonoff's fixed point theorem.

In §3 we consider a set X in a topological vector space E and a continuous linear transformation ϕ from E into itself. We are interested in conditions which will ensure the existence of a linear subspace contained in X and invariant under ϕ. Theorems 2 and 3 are results of this type. Less general results have been given in our earlier paper [3].

The results in §3 lend interesting geometric insight into certain known theorems on invariant linear subspaces with a special property for a particular class of linear transformations. As a first example, consider a linear transformation ϕ from the m-dimensional real vector space R^m into itself such that the matrix of ϕ in some basis of R^m is totally positive, i.e., all minors of the matrix are positive. For a fixed positive integer $n < m$, let X denote the set of all those $x \in R^m$ such that the number of variations of sign in the coordinates (with respect to the chosen basis) of x is at most $n-1$. The total positivity implies the variation-diminishing property, so that $\phi(X) \subset X$. It is easy to see that there

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exists a $(m-n)$-dimensional linear subspace H in R^m such that $X \cap (x+H)$ is compact and convex for every $x \in R^m$. Therefore, according to Theorem 3 below, there exists an n-dimensional linear subspace L such that $\phi(L) = L$ and the number of variations of sign in the coordinates of every point in L is at most $n-1$. This is a well-known property, discovered by Gantmacher and Krein, of totally positive linear transformations (see [4]).

Another known result related to Theorem 3 is the following theorem of Pontrjagin-Iohvidov-Krein [6, 8, 10]. Let E be the usual Hilbert space of infinite complex sequences $x = \{x_i\}$ with $\|x\| = (\sum_{i=1}^{\infty} |x_i|^2)^{1/2} < \infty$. Let n be a fixed positive integer and let $q_n(x) = \sum_{i=1}^{n} |x_i|^2 - \sum_{i=n+1}^{\infty} |x_i|^2$ for $x = \{x_i\} \in E$. If ϕ is a continuous linear transformation from E into itself, and if $q_n(x) \geqq 0$ implies $q_n(\phi(x)) \geqq q_n(x)$, then there exists an n-dimensional linear subspace L of E such that $\phi(L) \subset L$ and $q_n(x) \geqq 0$ for $x \in L$. This theorem is fundamental in the study of spectral properties of linear transformations in a Hilbert space with an indefinite inner product (see [7, 9]). A geometric reason for this theorem is again supplied by Theorem 3.

2. **Existence of invariant cross-sections.** For an arbitrary set F and a topological vector space E, we shall denote by E^F the topological vector space of all functions from F into E, with the topology of pointwise convergence. Thus E^F may be identified with the product space $\Pi_{y \in F} E_y$, where each E_y is a copy of E. When E is separated and locally convex, E^F is clearly also separated and locally convex.

THEOREM 1. *Let X be a set in a separated locally convex topological vector space E and Y a Hausdorff space. Let π be a continuous mapping from X onto Y, and ϕ a continuous mapping from X into E. Let Γ be an equicontinuous set of π-cross-sections such that Γ, when regarded as in E^Y, is a nonempty compact convex set in the topological vector space E^Y. If for every $\xi \in \Gamma$ there is an $\eta \in \Gamma$ such that $(\phi \circ \xi)(Y) \subset \eta(Y)$, then there exists a $\hat{\xi} \in \Gamma$ invariant under ϕ, i.e., $(\phi \circ \hat{\xi})(Y) \subset \hat{\xi}(Y)$.*

Proof. Consider any $\xi \in \Gamma$. By hypothesis, there is an $\eta \in \Gamma$ such that $(\phi \circ \xi)(Y) \subset \eta(Y)$. For each $y \in Y$, there is a $z \in Y$ with $(\phi \circ \xi)(y) = \eta(z)$. Then since $\pi \circ \eta$ is the identity mapping on Y, we have $(\phi \circ \xi)(y) = \eta(z) = (\eta \circ \pi \circ \eta)(z) = (\eta \circ \pi \circ \phi \circ \xi)(y)$. Thus, for each $\xi \in \Gamma$, there is an $\eta \in \Gamma$ such that

$$\phi \circ \xi = \eta \circ \pi \circ \phi \circ \xi. \tag{1}$$

Let $\xi_0 \in \Gamma$, a finite set $\{y_1, y_2, \cdots, y_n\} \subset Y$ and a neighborhood U of 0 in E be given. We choose a neighborhood V of 0 in E such that $\phi(x) - (\phi \circ \xi_0)(yi) \in U (1 \leqq i \leqq n)$ holds for all $x \in X$ satisfying $x - \xi_0(y_i) \in V (1 \leqq i \leqq n)$. Then $\xi \in \Gamma$ and $\xi(y_i) - \xi_0(y_i) \in V (1 \leqq i \leqq n)$ will imply $(\phi \circ \xi)(y_i) - (\phi \circ \xi_0)(y_i) \in U (1 \leqq i \leqq n)$. Hence the function $\xi \to \phi \circ \xi$ from Γ into E^Y is continuous on Γ.

Next, let $(\xi_0,\eta_0)\in\Gamma\times\Gamma$, a finite set $\{y_1,y_2,\cdots,y_n\}\subset Y$ and a neighborhood U of 0 in E be given. By equicontinuity of Γ, there exists for each $i=1,2,\cdots,n$, a neighborhood W_i of $(\pi\circ\phi\circ\xi_0)(y_i)$ in Y such that

$$\eta(y)-(\eta\circ\pi\circ\phi\circ\xi_0)(y_i)\in U \text{ for } y\in W_i \text{ and } \eta\in\Gamma.$$

For this neighborhood W_i of $(\pi\circ\phi\circ\xi_0)(y_i)$, we find a neighborhood U_1 of 0 in E such that for each $i=1,2,\cdots,n$:

$$(\pi\circ\phi\circ\xi)(y_i)\in W_i \text{ for all } \xi\in\Gamma \text{ satisfying } \xi(y_i)\in\xi_0(y_i)+U_1.$$

Let N_1 be the neighborhood of ξ_0 in Γ formed by all $\xi\in\Gamma$ satisfying

$$\xi(y_i)-\xi_0(y_i)\in U_1 \qquad (1\leqq i\leqq n).$$

Let N_2 be the neighborhood of η_0 in Γ formed by all $\eta\in\Gamma$ satisfying

$$(\eta\circ\pi\circ\phi\circ\xi_0)(y_i)-(\eta_0\circ\pi\circ\phi\circ\xi_0)(y_i)\in U \qquad (1\leqq i\leqq n).$$

Then we have

$$(\eta\circ\pi\circ\phi\circ\xi)(y_i)-(\eta\circ\pi\circ\phi\circ\xi_0)(y_i)\in U \text{ for } \xi\in N_1,\ \eta\in\Gamma \text{ and } 1\leqq i\leqq n,$$

and therefore

$$(\eta\circ\pi\circ\phi\circ\xi)(y_i)-(\eta_0\circ\pi\circ\phi\circ\xi_0)(y_i)\in U+U \text{ for } \xi\in N_1,\eta\in N_2 \text{ and } 1\leqq i\leqq n.$$

This proves that the function $(\xi,\eta)\to\eta\circ\pi\circ\phi\circ\xi$ from $\Gamma\times\Gamma$ into E^Y is continuous on $\Gamma\times\Gamma$.

Let Δ denote the set of all $(\xi,\eta)\in\Gamma\times\Gamma$ satisfying (1). For each $\xi\in\Gamma$, let $\Delta(\xi)=\{\eta\in\Gamma:(\xi,\eta)\in\Delta\}$. As we have seen at the beginning of the proof, $\Delta(\xi)$ is nonempty for every $\xi\in\Gamma$. Since Γ is convex, $\Delta(\xi)$ is convex. Since $(\xi,\eta)\to\phi\circ\xi-\eta\circ\pi\circ\phi\circ\xi$ is a continuous function from $\Gamma\times\Gamma$ into E^Y, Δ is closed in $\Gamma\times\Gamma$. Thus for every $\xi\in\Gamma$, $\Delta(\xi)$ is a nonempty closed convex subset of the compact convex set Γ.

The set-valued function $\xi\to\Delta(\xi)$ is upper semi-continuous on Γ in the following sense: for any $\xi_0\in\Gamma$ and any open set G in E^Y such that $\Delta(\xi_0)\subset G$, there is a neighborhood N_0 of ξ_0 in Γ such that $\Delta(\xi)\subset G$ for all $\xi\in N_0$. In fact, let $\mathscr{N}$ be the family of all neighborhoods of ξ_0 in Γ. If we denote $\bigcup_{\xi\in N}\Delta(\xi)$ by $\Delta(N)$, then since Δ is closed in $\Gamma\times\Gamma$, it is easy to see that $\Delta(\xi_0)=\bigcap_{N\in\mathscr{N}}\overline{\Delta(N)}$. If an open set G in E^Y contains $\Delta(\xi_0)$, then by compactness of Γ, there is a finite number of $N_1,N_2,\cdots,N_k\in\mathscr{N}$ such that $G\supset\bigcap_{j=1}^k\overline{\Delta(N_j)}$. Then $N_0=\bigcap_{j=1}^kN_j\in\mathscr{N}$ and $G\supset\Delta(N_0)$.

We now apply a generalization [1, 5] of a fixed-point theorem of Kakutani. For the upper semi-continuous set-valued function $\xi\to\Delta(\xi)$ defined on Γ, there exists a $\hat{\xi}\in\Gamma$ such that $\hat{\xi}\in\Delta(\hat{\xi})$. For this $\hat{\xi}$, we have $(\phi\circ\hat{\xi})(Y)=(\hat{\xi}\circ\pi\circ\phi\circ\hat{\xi})(Y)\subset\hat{\xi}(Y)$, which completes the proof.

Actually, Theorem 1 remains true, if the hypothesis on local convexity of the topological vector space E is replaced by the condition that every two distinct points of E may be separated by a continuous linear functional on E. This results from the following alternative proof.

Another proof of Theorem 1. As we have seen above, the proof of Theorem 1 amounts to showing the existence of a $\hat{\xi} \in \Gamma$ satisfying

$$\phi \circ \hat{\xi} = \hat{\xi} \circ \pi \circ \phi \circ \hat{\xi}. \tag{2}$$

As in the first proof, we shall still need the facts that the function $(\xi,\eta) \to \phi \circ \xi - \eta \circ \pi \circ \phi \circ \xi$ from $\Gamma \times \Gamma$ into E^Y is continuous on $\Gamma \times \Gamma$, and that for each $\xi \in \Gamma$ there is an $\eta \in \Gamma$ satisfying (1).

Instead of local convexity, we assume now that any two distinct points of E can be separated by a continuous linear functional on E. It follows that E^Y also has this property. For a continuous linear functional f on E^Y, let $\Phi(f)$ denote the set of all $\xi \in \Gamma$ satisfying $f(\phi \circ \xi - \xi \circ \pi \circ \phi \circ \xi) = 0$. The existence of a $\hat{\xi}$ satisfying (2) means $\bigcap_f \Phi(f) \neq \emptyset$, where the intersection is taken over all continuous linear functionals f on E^Y. Since $\Phi(f)$ is a closed subset of the compact set Γ, it suffices to prove that $\bigcap_{i=1}^n \Phi(f_i) \neq \emptyset$ for any finite number of continuous linear functionals f_i on E^Y.

Given any finite set $\{f_1, f_2, \cdots, f_n\}$ of continuous linear functionals on E^Y, consider the set Ψ formed by all $(\xi,\eta) \in \Gamma \times \Gamma$ satisfying

$$\sum_{i=1}^n |f_i(\phi \circ \xi - \xi \circ \pi \circ \phi \circ \xi)| \leqq \sum_{i=1}^n |f_i(\phi \circ \xi - \eta \circ \pi \circ \phi \circ \xi)|.$$

We observe that: (i) Γ is a nonempty compact convex set in E^Y and Ψ is a closed subset of $\Gamma \times \Gamma$; (ii) $(\xi,\xi) \in \Psi$ for each $\xi \in \Gamma$; (iii) for each $\xi \in \Gamma$, the set $\{\eta \in \Gamma : (\xi,\eta) \notin \Psi\}$ is convex (or empty). These facts (i)–(iii) imply [2, Lemma 4] the existence of a $\xi_1 \in \Gamma$ such that $(\xi_1, \eta) \in \Psi$ for all $\eta \in \Gamma$; this implication does not require local convexity of E^Y. For this ξ_1, we can find an $\eta_1 \in \Gamma$ such that $\phi \circ \xi_1 = \eta_1 \circ \pi \circ \phi \circ \xi_1$, which together with $(\xi_1, \eta_1) \in \Psi$ imply that $f_i(\phi \circ \xi_1 - \xi_1 \circ \pi \circ \phi \circ \xi_1) = 0$ for $1 \leqq i \leqq n$. Thus $\xi_1 \in \bigcap_{i=1}^n \Phi(f_i) \neq \emptyset$, which was to be proved.

COROLLARY 1. *Let X be a set in a separated locally convex topological vector space E. Let π be a continuous mapping from X onto a Hausdorff space Y such that the following conditions are fulfilled:*

(3) *For each $y \in Y$, $\pi^{-1}(y)$ is compact and convex.*

(4) *The set Γ_0 of all π-cross-sections is nonempty and equicontinuous on Y.*

Let ϕ be a continuous mapping from X into E. If for every $\xi \in \Gamma_0$ there is *an* $\eta \in \Gamma_0$ *such that* $(\phi \circ \xi)(Y) \subset \eta(Y)$, *then there exists a* $\hat{\xi} \in \Gamma_0$ *such that* $(\phi \circ \hat{\xi})(Y) \subset \hat{\xi}(Y)$.

Proof. By Theorem 1, it suffices to verify that the set Γ_0 of all π-cross-sections is a convex compact set in E^Y.

Let $\xi_1, \xi_2 \in \Gamma_0$ and let $\xi = c_1\xi_1 + c_2\xi_2$, where $c_1 \geqq 0$, $c_2 \geqq 0$ and $c_1 + c_2 = 1$. For each $y \in Y$, $\xi_1(y)$ and $\xi_2(y)$ are in the convex set $\pi^{-1}(y)$, so $\xi(y) \in \pi^{-1}(y)$. Hence $\xi(Y) \subset X$ and $\pi \circ \xi$ is the identity mapping on Y. As ξ is clearly continuous, we have $\xi \in \Gamma_0$. Thus Γ_0 is convex.

For each $y \in Y$, $\{\xi(y) : \xi \in \Gamma_0\}$ is contained in the compact set $\pi^{-1}(y)$. It follows that Γ_0 is relatively compact in E^Y. It remains to verify that Γ_0 is closed in E^Y. Let ξ_0 be in the closure of Γ_0 in E^Y. For each $y \in Y$, $\xi(y)$ is in the compact set $\pi^{-1}(y)$ for every $\xi \in \Gamma_0$; so we must have $\xi_0(y) \in \pi^{-1}(y)$. Thus $\xi_0(Y) \subset X$ and $\pi \circ \xi_0$ is the identity mapping on Y. That ξ_0 is continuous on Y follows from the equicontinuity of Γ_0 and the fact that ξ_0 is in the closure of Γ_0. Hence $\xi_0 \in \Gamma_0$. This verifies that Γ_0 is closed in E^Y and therefore is compact.

3. **Existence of invariant linear subspaces.** In this section we apply Theorem 1 to study invariant linear subspaces contained in a given set.

THEOREM 2. *Let E, F be two separated locally convex topological vector spaces, and let ρ be a continuous linear transformation from E onto F. Let X be a set in E having the following properties:*

(5) *For each $y \in F$, $\rho^{-1}(y) \cap X$ is compact and convex.*

(6) *There is a neighborhood W of 0 in F such that $\rho^{-1}(W) \cap X$ is bounded.*

(7) *X contains a linear subspace L (not necessarily closed) of E such that $\rho(L) = F$.*

Let Λ denote the family of all linear subspaces L of E such that $L \subset X$ and $\rho(L) = F$. If ϕ is a continuous mapping from X into E such that for every $L \in \Lambda$, $\phi(L)$ is contained in some $M \in \Lambda$, then there exists an $\hat{L} \in \Lambda$ with $\phi(\hat{L}) \subset \hat{L}$.

Proof. Let Γ denote the set of all $\xi \in E^F$ such that $\xi(F) \subset X$, ξ is linear and $\rho \circ \xi$ is the identity mapping on F. Then $\xi(F) \in \Lambda$ for every $\xi \in \Gamma$. By(5), $X \cap \operatorname{Ker} \rho$ is compact, so $L \cap \operatorname{Ker} \rho = \{0\}$ for every $L \in \Lambda$. Thus to each $L \in \Lambda$ there corresponds a unique $\xi \in \Gamma$ with $\xi(F) = L$. Hence $\Lambda = \{\xi(F) : \xi \in \Gamma\}$, and (7) means that Γ is nonempty.

Consider an arbitrary neighborhood U of 0 in E. By (6), $\rho^{-1}(W) \cap X$ is bounded, so there is an $r > 0$ such that $\rho^{-1}(W) \cap X \subset rU$. Then for every $\xi \in \Gamma$ we have $\xi(W) \subset \rho^{-1}(W) \cap X \subset rU$ or $\xi(r^{-1}W) \subset U$. As W is a neighborhood of 0 in F, this shows that the set Γ of linear transformations is equicontinuous.

Let $\xi_1, \xi_2 \in \Gamma$ and $\xi = c_1\xi_1 + c_2\xi_2$, where $c_1 \geqq 0$, $c_2 \geqq 0$, $c_1 + c_2 = 1$. Then for each $y \in F$, the points $\xi_1(y)$ and $\xi_2(y)$ are in the convex set $\rho^{-1}(y) \cap X$, so $\xi(y) \in \rho^{-1}(y) \cap X$. Hence $\xi(F) \subset X$ and $\rho \circ \xi$ is the identity mapping on F. As ξ is linear, we have $\xi \in \Gamma$. Thus Γ is a convex set in E^F.

Let ξ_0 be in the closure of Γ in E^F. Then ξ_0 is necessarily linear. For each $y \in F$, $\xi(y)$ is contained in the compact set $\rho^{-1}(y) \cap X$ for every $\xi \in \Gamma$. This implies $\xi_0(y) \in \rho^{-1}(y) \cap X$. Hence $\xi_0(F) \subset X$ and $\rho \circ \xi_0$ is the identity mapping on F, i.e., $\xi_0 \in \Gamma$. This shows that Γ is closed in E^F. For each $y \in F$, $\{\xi(y) : \xi \in \Gamma\}$ being contained in the compact set $\rho^{-1}(y) \cap X$, is relatively compact in E. It follows that Γ is relatively compact in E^F and therefore is compact.

If we denote by π the restriction of ρ on X, then $\pi(X) = F$. Γ is an equicontinuous set of π-cross-sections and it is a nonempty compact convex set in E^F.

Let $\xi \in \Gamma$ and $L = \xi(F)$. Then $L \in \Lambda$, so there is an $M \in \Lambda$ such that $\phi(L) \subset M$. Let $\eta \in \Gamma$ be such that $M = \eta(F)$. Then $(\phi \circ \xi)(F) = \phi(L) \subset \eta(F)$. By Theorem 1, there exists a $\hat{\xi} \in \Gamma$ such that $(\phi \circ \hat{\xi})(F) \subset \hat{\xi}(F)$. Then $\hat{L} = \hat{\xi}(F) \in \Lambda$ and $\phi(\hat{L}) \subset \hat{L}$, and the theorem is proved.

Corollary 2. *Let a normed vector space E be the direct sum $E = E_1 \oplus E_2$ of two closed linear subspaces E_1, E_2, of which E_2 is reflexive. For $x = y + z$ with $y \in E_1$, $z \in E_2$, let $q(x) = \| y \| - \| z \|$. Let Λ be the family of all those linear subspaces (not necessarily closed) L of E such that $\pi(L) = E_1$ and $q(x) \geqq 0$ for $x \in L$, where π denotes the projection from $E_1 \oplus E_2$ onto E_1. Let ϕ be a continuous linear transformation from E into itself. If, for any $L \in \Lambda$, there is an $M \in \Lambda$ with $\phi(L) \subset M$, then there exists an $\hat{L} \in \Lambda$ with $\phi(\hat{L}) \subset \hat{L}$.*

Proof. We use the weak topology of E, for which both π and ϕ remain continuous. Let $X = \{x \in E : q(x) \geqq 0\}$. For each $y \in E_1$, $\pi^{-1}(y) \cap X = \{y + z : z \in E_2$ and $\| z \| \leqq \| y \|\}$ is clearly convex; it is weakly compact since E_2 is reflexive. If $W = \{y \in E_1 : \| y \| \leqq 1\}$, then $\pi^{-1}(W) \cap X = \{y + z : y \in E_1, z \in E_2$ and $\| z \| \leqq \| y \| \leqq 1\}$ is bounded. Thus the result follows from Theorem 2.

Theorem 3. *Let E be a separated locally convex topological vector space. Let n be a positive integer and X a set in E having the following properties:*

(8) *There exists a closed linear subspace H in E of codimension n such that $X \cap (x + H)$ is compact and convex for every $x \in E$.*

(9) *X contains an n-dimensional linear subspace of E.*

Let ϕ be a continuous linear transformation from E into itself such that:

(10) $\phi(X) \subset X$.

(11) *No 1-dimensional linear subspace of E is contained in $X \cap \operatorname{Ker} \phi$.*

Then there exists an n-dimensional linear subspace L of E such that $L \subset X$ and $\phi(L) = L$.

Proof. Observe first that in case F is finite dimensional, Theorem 2 remains valid without hypothesis (6). In the proof of Theorem 2, hypothesis (6) was used only in establishing the equicontinuity of Γ. (For the notation, see Theorem 2 and its proof.) In case F is of finite dimension n, the equicontinuity of Γ is a consequence of (5). In fact, let $\{e_1, e_2, \cdots, e_n\}$ be a basis of F, and let U be an arbitrary balanced convex neighborhood of 0 in E. Since $\rho^{-1}(e_i) \cap X$ is compact, there is an $r > 0$ such that $r(\rho^{-1}(e_i) \cap X) \subset U$ for $1 \leqq i \leqq n$. Then $\xi(e_i) \in r^{-1} U$ for all $\xi \in \Gamma$ and $1 \leqq i \leqq n$. As U is balanced and convex, we have $\xi(y) = \sum_{i=1} c_i \xi(e_i) \in U$ for all $\xi \in \Gamma$ and all $y = \sum_{i=1}^n c_i e_i$ satisfying $\sum_{i=1}^n |c_i| \leqq r$. Thus Γ is equicontinuous.

Now under the hypothesis of the present theorem, let ρ denote the canonical mapping from E onto E/H, and let $F = E/H$. Then $\dim F = n$. Let Λ denote the family of all n-dimensional linear subspaces of E contained in X. By (10) and (11), we have $\phi(L) \subset X$ and $\dim \phi\ (L) = n$ for $L \in \Lambda$. Thus $\phi(L) \in \Lambda$ for $L \in \Lambda$. By (8), $X \cap \operatorname{Ker} \rho = X \cap H$ is compact, so $L \cap \operatorname{Ker} \rho = \{0\}$ for $L \in \Lambda$. Hence Λ is precisely the family of all linear subspaces L of E such that $L \subset X$ and $\rho(L) = F$. All hypotheses except (6) of Theorem 2 are satisfied. Since F is finite dimensional, the present theorem follows.

The following corollary has been given in [3]. It follows from Theorem 3 in the same manner that Corollary 2 follows from Theorem 2.

COROLLARY 3. *Let a Banach space E be the direct sum $E = E_1 \oplus E_2$ of two linear subspaces, of which E_1 is of finite dimension n and E_2 is reflexive. For $x = y + z$ with $y \in E_1$, $z \in E_2$, let $q(x) = \| y \| - \| z \|$. Let ϕ be a continuous linear transformation from E into itself. If $x \neq 0$ and $q(x) \geqq 0$ imply $\phi(x) \neq 0$ and $q(\phi(x)) \geqq 0$, then there exists an n-dimensional linear subspace L of E such that $\phi(L) = L$ and $q(x) \geqq 0$ for all $x \in L$.*

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